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LTL over finite words can be exponentially more succinct than pure-past LTL, and *vice versa*

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Contributions

- $\mathsf{LTL_f}$: Linear Temporal Logic over *finite words*
  - only future operators
- pLTL: *pure past* LTL
  - only past operators
- $\mathsf{LTL_f}$ and pLTL are expressively equivalent [1]
- To the best of our knowledge, there is no systematic study of their succinctness.

[1] Orna Lichtenstein, Amir Pnueli, and Lenore Zuck (1985). "The glory of the past". In: *Workshop on Logic of Programs*. Springer, pp. 196–218. DOI: 10.1007/3-540-15648-8_16

Contributions

Contributions:

1. $\mathsf{LTL_f}$ can be exponentially more succinct than pLTL
2. pLTL can be exponentially more succinct than $\mathsf{LTL_f}$

$\Rightarrow$ incomparability result

Contributions

- SafetyLTL: the fragment of LTL in NNF with only universal temporal operators (X, G, R)
- G(pLTL): all formulas of the form $\mathsf{G}(\alpha)$ where $\alpha \in$ pLTL
  - canonical form for SafetyLTL
- All safety properties expressible in LTL can be defined in SafetyLTL and G(pLTL) …but at which cost?
- The succinctness of SafetyLTL w.r.t. G(pLTL) has not been studied.
- The same also holds for coSafetyLTL and F(pLTL)

Contributions

Contributions:

3. G(pLTL) can be exponentially more succinct than SafetyLTL
4. F(pLTL) can be exponentially more succinct than coSafetyLTL

⇒ this confirms the conjecture formulated in [1], derived from the complexity gap between the realizability problem of SafetyLTL, which is 2EXPTIME-complete, and G(pLTL), which is EXPTIME-complete;

Logic	Complexity of realiz.
SafetyLTL	2EXPTIME-c
G(pLTL)	EXPTIME-c

[1] Alessandro Artale et al. (2023). "Complexity of Safety and coSafety Fragments of Linear Temporal Logic". In: *Proc. of the 36th AAAI Conf. on Artificial Intelligence*. AAAI Press

BACKGROUND

Linear Temporal Logic with Past (LTL+P, for short) is a *modal* logic.

- it extends classical *propositional* logic
- temporal *operators* are used to talk about how propositions change over time

Let $\mathcal{AP} \coloneqq \{p, q, r, \ldots\}$ be a set of *atomic propositions*. The syntax of LTL+P is defined as follows:

$$\phi \coloneqq p \mid \neg p \mid \phi \vee \phi \mid \phi \wedge \phi \qquad \text{Boolean Modalities}$$
$$\mid \mathsf{X}\phi \mid \widetilde{\mathsf{X}}\phi \mid \phi \,\mathsf{U}\, \phi \mid \phi \,\mathsf{R}\, \phi \qquad \text{Future Temporal Modalities}$$
$$\mid \mathsf{Y}\phi \mid \widetilde{\mathsf{Y}}\phi \mid \phi \,\mathsf{S}\, \phi \mid \phi \,\mathsf{T}\, \phi \qquad \text{Past Temporal Modalities}$$

where $p \in \mathcal{AP}$. Negation only in front of atomic propositions.

- X is called *tomorrow* (or *next*)
- $\widetilde{\mathsf{X}}$ is called *weak tomorrow*
- U is called *until*
- R is called *release*

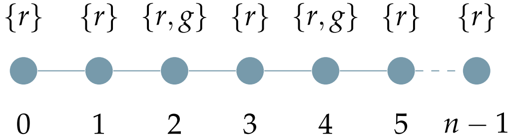
- Y is called *yesterday* (or *previous*)
- $\widetilde{\mathsf{Y}}$ is called *weak yesterday*
- S is called *since*
- T is called *triggers*

Linear Temporal Logic with Past
LTL+P Semantics

- A *state* is a member of $2^{\mathcal{AP}}$.
- Finite word / state sequence / trace is a finite (nonempty) sequence of states:

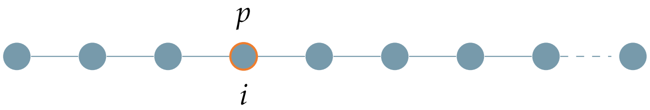
$$\sigma \in (2^{\mathcal{AP}})^+$$

$$\mathcal{AP} := \{r, g\}$$



We say that σ satisfies at position i the LTL+P formula ϕ, written $\sigma, i \models \phi$, iff:

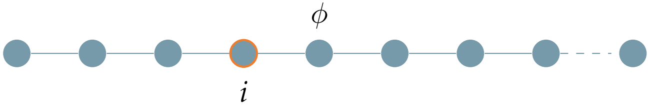
- $\sigma, i \models p$ iff $p \in \sigma_i$



p holds at position i

We say that σ satisfies at position i the LTL+P formula ϕ, written $\sigma, i \models \phi$, iff:

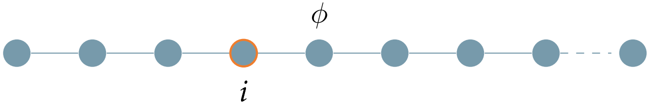
- $\sigma, i \models \mathsf{X}\phi$ iff $i < |\sigma| - 1$ and $\sigma, i+1 \models \phi$



ϕ holds at the *next* position of i

We say that σ satisfies at position i the LTL+P formula ϕ, written $\sigma, i \models \phi$, iff:

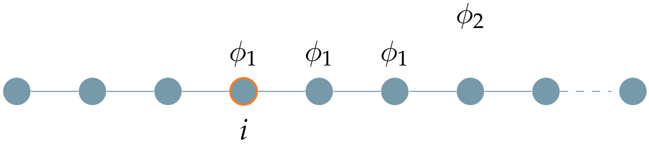
- $\sigma, i \models \widetilde{\mathsf{X}}\phi$ iff $i = |\sigma| - 1$ or $\sigma, i+1 \models \phi$



ϕ holds at the *next* position of i, *if any*

We say that σ satisfies at position i the LTL+P formula ϕ, written $\sigma, i \models \phi$, iff:

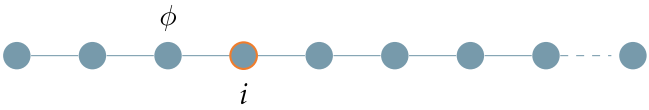
- $\sigma, i \models \phi_1 \mathsf{U} \phi_2$ iff $\exists i \leq j < |\sigma| \,.\, \sigma, j \models \phi_2$ and $\forall i \leq k < j \,.\, \sigma, k \models \phi_1$



ϕ_1 holds *until* ϕ_2 holds

We say that σ satisfies at position i the LTL+P formula ϕ, written $\sigma, i \models \phi$, iff:

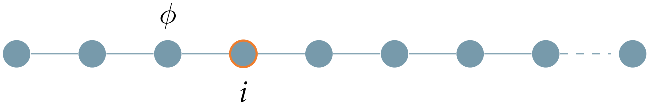
- $\sigma, i \models \mathsf{Y}\phi$ iff $i > 0$ and $\sigma, i-1 \models \phi$



position i has a predecessor and ϕ holds at the *previous* position of i

We say that σ satisfies at position i the LTL+P formula ϕ, written $\sigma, i \models \phi$, iff:

- $\sigma, i \models \widetilde{\mathsf{Y}}\phi$ iff $i = 0$ or $\sigma, i-1 \models \phi$

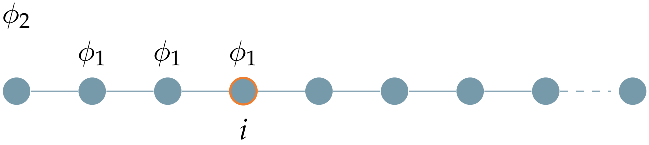


ϕ holds at the *previous* position of i, *if any*

Note: $\sigma, i \models \widetilde{\mathsf{Y}}\bot$ iff $i = 0$.

We say that σ satisfies at position i the LTL+P formula ϕ, written $\sigma, i \models \phi$, iff:

- $\sigma, i \models \phi_1 \mathsf{S}\, \phi_2$ iff $\exists j \leq i \,.\, \sigma, j \models \phi_2$ and $\forall j < k \leq i \,.\, \sigma, k \models \phi_1$



ϕ_1 holds *since* ϕ_2 held

Notation

- We say that two formulas $\phi, \psi \in \mathsf{LTL{+}P}$ are *equivalent*, written $\phi \equiv \psi$, when, for all $\sigma \in (2^\Sigma)^\omega$, it holds that $\sigma \models \phi$ if and only if $\sigma \models \psi$.
- The size of ϕ, denoted with $|\phi|$, is the number nodes of its parse tree.

Definition ($\mathsf{LTL_f}$)

We denote with $\mathsf{LTL_f}$ the logic $\mathsf{LTL{+}P}$ devoid of past temporal operators and interpreted over finite traces.

The pure past fragment of LTL+P

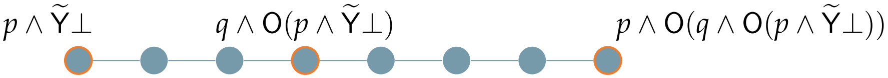
Definition (Pure-past LTL)

Pure-past LTL (pLTL, for short) is the set of LTL+P formulas *devoid* of future operators.

Formulas of pLTL are naturally interpreted on the *last* position of a *finite trace*.

Example:

$p \wedge \mathsf{O}(q \wedge \mathsf{O}(p \wedge \widetilde{\mathsf{Y}}\bot))$



The F(pLTL) and G(pLTL) fragments

F(pLTL)

Definition

$\phi := \mathsf{F}(\alpha)$, where $\alpha \in \mathsf{pLTL}$, that is α is a pure-past LTL formula.

Example:

$\mathsf{F}(q \wedge \widetilde{\mathsf{Y}}\mathsf{H}p)$

G(pLTL)

Definition

$\phi := \mathsf{G}(\alpha)$, where $\alpha \in \mathsf{pLTL}$, that is α is a pure-past LTL formula.

Example:

$\mathsf{G}(\widetilde{\mathsf{Y}}\widetilde{\mathsf{Y}}r \rightarrow g)$

The glory of the past

The *pure past fragments* have several important properties:

1. The conversion of $\mathsf{LTL_f}$ into DFA is doubly exponential (lower bound).

Theorem

For each pLTL *formula of size n, there exists an equivalent* DFA *of size* $2^{\mathcal{O}(n)}$.

This conversion can be done fully symbolically.

Reference

- Giuseppe De Giacomo et al. (2021). "Pure-past linear temporal and dynamic logic on finite traces". In: *Proceedings of the Twenty-Ninth International Conference on International Joint Conferences on Artificial Intelligence*, pp. 4959–4965
- Alessandro Cimatti et al. (2021). "Extended bounded response LTL: a new safety fragment for efficient reactive synthesis". In: *Formal Methods in System Design*, 1–49 (published online on November 18, 2021, doi: 10.1007/s10703-021-00383–3)

The glory of the past

The *pure past fragments* have several important properties:

2. The *reactive synthesis* problem for LTL and $\mathsf{LTL_f}$ is 2EXPTIME-complete.

Theorem

The reactive synthesis problem for pLTL, F(pLTL), *and* G(pLTL) *is* EXPTIME*-complete.*

Reference

Alessandro Artale et al. (2023). "Complexity of Safety and coSafety Fragments of Linear Temporal Logic". In: *Proc. of the 36th AAAI Conf. on Artificial Intelligence.* AAAI Press

SUCCINCTNESS

Succinctness

Definition

Given two linear-time temporal logics $\mathbb{L}$ and $\mathbb{L}'$, we say that *$\mathbb{L}$ can be exponentially more succinct than $\mathbb{L}'$ over finite trace* iff there exists an alphabet Σ and a family of languages $\{\mathcal{L}_n\}_{n>0} \subseteq (2^{\Sigma})^*$ such that, for any $n > 0$:

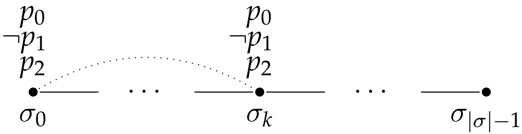
- there exists a formula $\phi \in \mathbb{L}$ over Σ such that its language over finite traces is $\mathcal{L}_n$ and $|\phi| \in \mathcal{O}(n)$; and
- for all formulas $\phi' \in \mathbb{L}'$ over Σ, if the language of ϕ' over finite traces is $\mathcal{L}_n$, then $|\phi'| \in 2^{\Omega(n)}$.

pLTL can be exponentially more succinct than LTLf

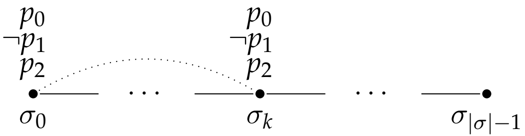
pLTL can be exp. more succinct than LTLf

Let $\Sigma = \{p_0, p_1, \ldots, p_n\}$ be a finite set of proposition letters. Consider the following family of languages over the alphabet 2^Σ, where $n > 0$.

$$A_n \coloneqq \{\sigma \in (2^\Sigma)^+ \mid \exists k > 0 \;.\; (\bigwedge_{i=0}^{n} (p_i \in \sigma_k \leftrightarrow p_i \in \sigma_0))\}$$



pLTL can be exp. more succinct than LTLf



We shall prove that all formulas of $\mathsf{LTL_f}$ defining A_n are at least of size exponential in n. Conversely:

Lemma

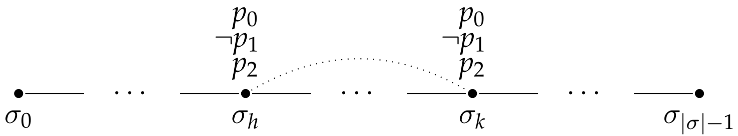
For any $n > 0$, there exists a formula $\phi \in \mathsf{pLTL}$ such that $\mathcal{L}(\phi) = A_n$ and $|\phi| \in \mathcal{O}(n)$.

Proof.

$\mathsf{O}(\bigwedge_{i=0}^{n}(p_i \leftrightarrow \mathsf{YO}(\widetilde{\mathsf{Y}}\bot \wedge p_i)))$. Note the crucial role of $\widetilde{\mathsf{Y}}\bot$ for hooking the initial state of a word. □

To prove that A_n is not expressible in $\mathsf{LTL_f}$ with formulas of size less than $2^{\Omega(n)}$ (for any $n > 0$), we make use of an auxiliary family of languages. For each $n > 0$, we define the language B_n over the alphabet 2^Σ with $\Sigma = \{p_0, p_1, \ldots, p_n\}$ as follows:

$$B_n \coloneqq \{\sigma \in (2^\Sigma)^+ \mid \exists h \geq 0 \,.\, \exists k > h \,.\, (\bigwedge_{i=0}^{n}(p_i \in \sigma_k \leftrightarrow p_i \in \sigma_h))\}$$



We now show that, if A_n was expressible in $\mathsf{LTL_f}$ in space less than exponential in n, then the property B_n would be expressible in $\mathsf{LTL_f}$ in space less than exponential as well.

Lemma

If there exists a formula of $\mathsf{LTL_f}$ for A_n of size less than exponential in n, then there exists a formula of $\mathsf{LTL_f}$ for B_n of size less than exponential in n.

Proof.

Let ψ_{A_n} be a formula of $\mathsf{LTL_f}$ for A_n of size less than exponential in n. Consider the formula $\mathsf{F}(\psi_{\mathsf{A}_n})$: we prove that its language is exactly B_n. For any $\sigma \in (2^\Sigma)^+$ and for any $n > 0$, it holds that $\sigma \models \mathsf{F}(\psi_{\mathsf{A}_n})$ iff $\exists k \geq 0 \ . \ \sigma_{[k,-]} \models \psi_{A_n}$, where $\sigma_{[k,-]}$ is the suffix of σ starting from i. This means: $\exists k \geq 0 \ . \ \exists h > k \ . \ (\bigwedge_{i=0}^{n} (\sigma_k \models p_i \leftrightarrow \sigma_h \models p_i))$. Equivalently, $\sigma \in B_n$. Moreover $\mathsf{F}(\psi_{\mathsf{A}_n})$ belongs to $\mathsf{LTL_f}$ and it is of size less than exponential in n. □

We now show that, if A_n was expressible in $\mathsf{LTL_f}$ in space less than exponential in n, then the property B_n would be expressible in $\mathsf{LTL_f}$ in space less than exponential as well.

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pLTL can be exp. more succinct than LTLf

Lemma

For any $n > 0$ and for any NFA $\mathcal{A}$ *over the alphabet* 2^{Σ}*, if* $\mathcal{L}(\mathcal{A}) = B_n$ *then* $|\mathcal{A}| \in 2^{2^{\Omega(n)}}$.

Proof.

Let $n > 0$. We fix a permutation $\langle a_0, \ldots, a_{2^n-1} \rangle$ of the 2^n subsets of $\{p_1, \ldots, p_n\}$ (note that this set does not include the proposition letter $p_0 \in \Sigma$).
Let $K \subseteq \{0, \ldots, 2^n - 1\}$ and let $\overline{K}$ be the complement set of K. We define b_i^K in this way:

$$b_i^K := \begin{cases} a_i & \text{if } i \in \overline{K} \\ a_i \cup \{p_0\} & \text{otherwise} \end{cases}$$

We define σ_K as the sequence $\langle b_0^K, b_1^K, \ldots, b_{2^n-1}^K \rangle$.

pLTL can be exp. more succinct than LTLf

Lemma

For any $n > 0$ *and for any* NFA $\mathcal{A}$ *over the alphabet* 2^Σ*, if* $\mathcal{L}(\mathcal{A}) = B_n$ *then* $|\mathcal{A}| \in 2^{2^{\Omega(n)}}$.

Proof.

Suppose by contradiction that there exists an NFA $\mathcal{A}$ for B_n of size less than doubly exponential in n. Consider the words:

1. $\sigma_K \cdot \sigma_K$
2. $\sigma_{\overline{K}} \cdot \sigma_{\overline{K}}$
3. $\sigma_K \cdot \sigma_{\overline{K}}$

pLTL can be exp. more succinct than LTLf

Lemma

For any $n > 0$ *and for any* NFA $\mathcal{A}$ *over the alphabet* 2^Σ*, if* $\mathcal{L}(\mathcal{A}) = B_n$ *then* $|\mathcal{A}| \in 2^{2^{\Omega(n)}}$.

Proof.

Suppose by contradiction that there exists an NFA $\mathcal{A}$ for B_n of size less than doubly exponential in n. Consider the words:

1. $\sigma_K \cdot \sigma_K$ is accepted by $\mathcal{A}$;
2. $\sigma_{\overline{K}} \cdot \sigma_{\overline{K}}$ is accepted by $\mathcal{A}$;
3. $\sigma_K \cdot \sigma_{\overline{K}}$ is *not* accepted by $\mathcal{A}$.

pLTL can be exp. more succinct than LTLf

Lemma

For any $n > 0$ and for any NFA $\mathcal{A}$ *over the alphabet* 2^Σ*, if* $\mathcal{L}(\mathcal{A}) = B_n$ *then* $|\mathcal{A}| \in 2^{2^{\Omega(n)}}$.

Proof.

- Let $\pi := \langle \pi_0, \ldots, \pi_{2^n-1} \rangle \cdot \langle \pi_{2^n}, \ldots, \pi_{2^{n+1}-1}, \ldots \rangle$ be any *accepting* run of $\mathcal{A}$ over the word $\sigma_K \cdot \sigma_K$.
- Let $\pi' := \langle \pi'_0, \ldots, \pi'_{2^n-1} \rangle \cdot \langle \pi'_{2^n}, \ldots, \pi'_{2^{n+1}-1}, \ldots \rangle$ be any *accepting* run of $\mathcal{A}$ over the word $\sigma_{\overline{K}} \cdot \sigma_{\overline{K}}$.

Suppose that $\pi_{2^n-1} = \pi'_{2^n-1}$. Let π'' be the sequence obtained by appending the suffix of π' starting from its 2^n-th state to the prefix of π of length $2^n - 1$, *i.e.*:

$$\pi'' := \langle \pi_0, \ldots, \pi_{2^n-1}, \pi'_{2^n}, \ldots, \pi'_{2^{n+1}-1}, \ldots \rangle$$

pLTL can be exp. more succinct than LTLf

Lemma

For any $n > 0$ *and for any* NFA $\mathcal{A}$ *over the alphabet* 2^{Σ}, *if* $\mathcal{L}(\mathcal{A}) = B_n$ *then* $|\mathcal{A}| \in 2^{2^{\Omega(n)}}$.

Proof.

$$\pi'' := \langle \pi_0, \dots, \pi_{2^n - 1}, \pi'_{2^n}, \dots, \pi'_{2^{n+1}-1}, \dots \rangle$$

By construction, π'' is an *accepting run* of the automaton $\mathcal{A}$ over the word $\sigma_K \cdot \sigma_{\overline{K}}$, which is a contradiction.
Therefore, the states at position 2^n-th of π and π' must be distinct. This means that the automaton $\mathcal{A}$ has to contain at least a state for choice of $K \subseteq \{0, \dots, 2^n - 1\}$.
Since there are 2^{2^n} of such possible choices, this means that $\mathcal{A}$ has to contain at least $2^{2^{\Omega(n)}}$ states. □

pLTL can be exp. more succinct than LTLf

Proposition ([1])

For any formula ϕ *of* $\mathsf{LTL_f}$ *of size* n*, there exists an* NFA $\mathcal{A}$ *such that* $\mathcal{L}(\phi) = \mathcal{L}(\mathcal{A})$ *and* $|\mathcal{A}| \in 2^{\mathcal{O}(n)}$.

Lemma

For any formula $\phi \in \mathsf{LTL_f}$*, if* $\mathcal{L}(\phi) = B_n$ *then* $|\phi| \in 2^{\Omega(n)}$.

[1] Giuseppe De Giacomo and Moshe Y. Vardi (2013). "Linear Temporal Logic and Linear Dynamic Logic on Finite Traces". In: *Proceedings of the 23rd International Joint Conference on Artificial Intelligence*. Ed. by Francesca Rossi. IJCAI/AAAI, pp. 854–860

pLTL can be exp. more succinct than LTLf

Lemma

If there exists a formula of $\mathsf{LTL_f}$ for A_n of size less than exponential in n, then there exists a formula of $\mathsf{LTL_f}$ for B_n of size less than exponential in n.

Theorem

For any $n > 0$ *and for any formula* $\phi \in \mathsf{LTL_f}$*, if* $\mathcal{L}(\phi) = A_n$ *then* $|\phi| \in 2^{\Omega(n)}$.

Corollary

pLTL *can be exponentially more succinct than* $\mathsf{LTL_f}$.

LTLf can be exponentially more succinct than pLTL

Definition (Reverse Language)

Given an alphabet Σ and a language $\mathcal{L} \subseteq (2^\Sigma)^+$ of finite words over 2^Σ, we define the *reverse language* of $\mathcal{L}$ as the set:

$$\mathcal{L}^- = \{\sigma' \in (2^\Sigma)^+ \mid \sigma'_i = \sigma_{n-i}, \text{ for } \sigma = \sigma_0 \ldots \sigma_n \in \mathcal{L} \text{ and } 0 \leq i \leq n\}.$$

Definition (Reverse Logics)

Given two linear-time temporal logics $\mathbb{L}$ and $\mathbb{L}^-$, we say that $\mathbb{L}^-$ is a *reverse logic* of $\mathbb{L}$ iff:

1. $\forall \phi \in \mathbb{L} \,.\, \exists \phi' \in \mathbb{L}^-$ such that $\mathcal{L}(\phi) = \mathcal{L}(\phi')^-$ and $|\phi'| = |\phi|$;
2. $\forall \phi' \in \mathbb{L}^- \,.\, \exists \phi \in \mathbb{L}$ such that $\mathcal{L}(\phi') = \mathcal{L}(\phi)^-$ and $|\phi| = |\phi'|$.

LTLf can be exp. more succinct than pLTL

Example

Consider the logic pLTL and any formula $\phi \in$ pLTL. By replacing in ϕ the temporal operators Y, $\widetilde{\mathsf{Y}}$, S, and T with X, $\widetilde{\mathsf{X}}$, U, and R, respectively, one obtains a formula ϕ' such that:

1. it belongs to LTL_f;
2. its size is $|\phi|$;
3. it is such that $\mathcal{L}(\phi) = \mathcal{L}(\phi')^-$.

Therefore, LTL_f is a reverse logic of pLTL, and *vice versa*.

LTLf can be exp. more succinct than pLTL

Lemma (Reverse Lemma)

For any two linear-time temporal logics $\mathbb{L}$ and $\mathbb{L}^-$ such that $\mathbb{L}$ is a reverse logic of $\mathbb{L}^-$, if a language $\mathcal{L}$ with a compact definition in $\mathbb{L}$ is not succinctly definable in $\mathbb{L}^-$, then $\mathcal{L}^-$ (i.e., the reverse language of $\mathcal{L}$) is compactly definable in $\mathbb{L}^-$, but its definitions exponentially blow-up in $\mathbb{L}$.

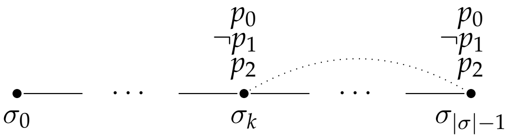
Corollary

For any two linear-time temporal logics $\mathbb{L}$ and $\mathbb{L}^-$ such that $\mathbb{L}$ is a reverse logic of $\mathbb{L}^-$, if $\mathbb{L}$ can be exponentially more succinct than $\mathbb{L}^-$, then $\mathbb{L}^-$ can be exponentially more succinct than $\mathbb{L}$.

LTLf can be exp. more succinct than pLTL

From the Reverse Lemma, we obtain a concrete family of languages that are definable with $\mathsf{LTL_f}$ formulas of polynomial size but such that any pLTL formula for them requires at least an exponential amount of space.

$$A_n^- := \{\sigma \in (2^\Sigma)^+ \mid \exists k < |\sigma| - 1 \;.\; (\bigwedge_{i=0}^{n} (p_i \in \sigma_k \leftrightarrow p_i \in \sigma_{|\sigma|-1}))\}$$



LTLf can be exp. more succinct than pLTL

For each $n > 0$, A_n^- can be expressed in $\mathsf{LTL_f}$ in space linear in n with the formula

$$\mathsf{F}(\bigwedge_{i=0}^{n}(p_i \leftrightarrow \mathsf{XF}(\widetilde{\mathsf{X}}\bot \wedge p_i))).$$

However, since $\mathsf{LTL_f}$ is a reverse logic of pLTL, by the Reverse Lemma every formula of pLTL for A_n^- requires an amount of space at least exponential in n.

Theorem

For any $n > 0$ *and for any formula* $\phi \in \mathsf{pLTL}$, *if* $\mathcal{L}(\phi) = A_n^-$ *then* $|\phi| \in 2^{\Omega(n)}$.

Corollary

$\mathsf{LTL_f}$ *can be exponentially more succinct than* pLTL.

Meaningful implications

1. $\mathsf{LTL_f}$ and pLTL:
   - syntax: incomparable
   - semantics: equivalent
   - succinctness: incomparable
2. Confirms the conjecture in [1]:
   - $\mathsf{LTL_f}$ realizability: 2EXPTIME-c
   - pLTL realizability: EXPTIME-c
   - succinctness gap
3. Succinctness can help in choosing the right formalism to express a property
4. The most efficient translation of $\mathsf{LTL_f}$ into pLTL is *triply exponential*
   - *any* translation from $\mathsf{LTL_f}$ to pLTL (not only the above one) has at least an exponential lower bound.

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[1] Alessandro Artale et al. (2023). "Complexity of Safety and coSafety Fragments of Linear Temporal Logic". In: *Proc. of the 36th AAAI Conf. on Artificial Intelligence*. AAAI Press

Succinctness of safety and cosafety fragments of LTL

SafetyLTL

Definition

$\phi := p \mid \neg p \mid \phi \land \phi \mid \phi \lor \phi \mid \mathsf{X}\phi \mid \mathsf{G}\phi \mid \phi \,\mathsf{R}\, \phi$

Example:

$\mathsf{G}(r \rightarrow \mathsf{X}\mathsf{X}g)$

G(pLTL)

Definition

$\phi := \mathsf{G}(\alpha)$, where $\alpha \in$ pLTL, that is α is a pure-past LTL formula.

Example:

$\mathsf{G}(\widetilde{\mathsf{Y}}\widetilde{\mathsf{Y}}r \rightarrow g)$

SafetyLTL and G(pLTL) are expressively equivalent.

G(pLTL) can be exp. more succinct than SafetyLTL

Theorem

G(pLTL) *can be exponentially more succinct than* SafetyLTL.

It derives from Markey's proof that LTL+P can be exponentially more succinct than LTL.

Reference

Nicolas Markey (2003). "Temporal logic with past is exponentially more succinct". In: *Bull. EATCS* 79, pp. 122–128

Theorem

G(pLTL) *can be exponentially more succinct than* SafetyLTL.

Let $\Sigma = \{p_0, \ldots, p_n\}$. Consider the family of languages M_n over the alphabet 2^Σ:

$$M_n := \{\sigma \in (2^\Sigma)^\omega \mid \forall k > 0(\forall i, 1 \leq i \leq n\, (p_i \in \sigma_k \leftrightarrow p_i \in \sigma_0) \leftrightarrow (p_0 \in \sigma_k \leftrightarrow p_0 \in \sigma_0))\}$$

Lemma (Markey)

For any $n > 0$, any formula of LTL *expressing M_n is at least of size exponential in n.*

Corollary

For any $n > 0$, any formula of SafetyLTL *expressing M_n is at least of size exponential in n.*

Theorem

G(pLTL) *can be exponentially more succinct than* SafetyLTL.

However, for each $n > 0$, there is a formula in G(pLTL) of size linear in n expressing M_n, such as the following:

$$\mathsf{G}((\bigwedge_{i=1}^{n}(p_i \leftrightarrow \mathsf{O}(\widetilde{\mathsf{Y}}\bot \wedge p_i))) \leftrightarrow (p_0 \leftrightarrow \mathsf{O}(\widetilde{\mathsf{Y}}\bot \wedge p_0)).)$$

Theorem

G(pLTL) *can be exponentially more succinct than* SafetyLTL.

Lemma (Duality Lemma)

For any linear-time temporal logics $\mathbb{L}$ and $\mathbb{L}'$, if $\mathbb{L}$ can be exponentially more succinct than $\mathbb{L}'$, then $\overline{\mathbb{L}}$ can be exponentially more succinct than $\overline{\mathbb{L}'}$, where $\overline{\mathbb{L}}$ (resp., $\overline{\mathbb{L}'}$) is a dual logic of $\mathbb{L}$ (resp., $\mathbb{L}'$).

Theorem

F(pLTL) *can be exponentially more succinct than* coSafetyLTL.

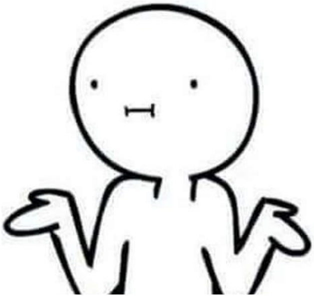
Succinctness of (co)safety fragments

- G(pLTL) can be exponentially more succinct than SafetyLTL
- F(pLTL) can be exponentially more succinct than coSafetyLTL

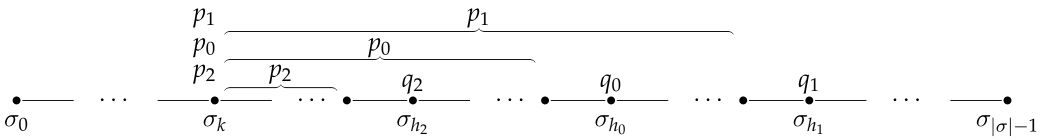
Succinctness of (co)safety fragments

- G(pLTL) can be exponentially more succinct than SafetyLTL
- F(pLTL) can be exponentially more succinct than coSafetyLTL

Does the viceversa hold as well?



For any $n > 0$, we define C_n as the language of the formula $\mathsf{F}(\bigwedge_{i=1}^{n}(p_i \mathbin{\mathsf{U}} q_i))$.



Conjecture

For any $n > 0$, the language C_n is not expressible in $\mathsf{F}(\mathsf{pLTL})$ with a formula of size less than $n!$.

CONCLUSIONS

Conclusions

1. Incomparability between the succinctness of $\mathsf{LTL_f}$ and of pLTL
   1. Family A_n for proving that pLTL can be exp. more succinct than $\mathsf{LTL_f}$
   2. Reverse Lemma for proving that $\mathsf{LTL_f}$ can be exp. more succinct than LTL
2. G(pLTL) can be exp. more succinct than SafetyLTL
3. F(pLTL) can be exp. more succinct than coSafetyLTL

Conclusions

1. The study of the maximal fragment of $\mathsf{LTL_f}$ that does not incur in the exponential blow-up in the translation into pLTL.
2. Proving the $n!$ lower bound for the succinctness of $\mathsf{coSafetyLTL}$ w.r.t. $\mathsf{F}(\mathsf{pLTL})$
   - these techniques does not work
   - Adler-Immermann games
3. Finally, while we know that the lower bound between the translation of $\mathsf{LTL_f}$ into pLTL is at least exponential, we have an upper bound which is triply exponential. The possibility of tighter lower bounds, or more efficient algorithms for this problem, is worth investigating.

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