A Sound and Complete Tableau System for Fuzzy HS

Willem Conradie
Emilio Muñoz-Velasco
Riccardo Monego
Guido Sciavicco guido.sciavicco@unife.it
Ionel Eduard Stan

Applied Computational Logic and Artificial Intelligence Lab (ACLAI-Lab),
Department of Mathematics and Computer Science,
University of Ferrara, Italy

Summary

Symbolic modern artificial intelligence is based on a set of algorithms and methods that are designed to extract knowledge from data in the form of logical formulas.

Summary

Symbolic modern artificial intelligence is based on a set of algorithms and methods that are designed to extract knowledge from data in the form of logical formulas. In the case of temporal data, in particular, interval temporal logic are used to describe patterns with quite a lot of expressive power.

Summary

Symbolic modern artificial intelligence is based on a set of algorithms and methods that are designed to extract knowledge from data in the form of logical formulas. In the case of temporal data, in particular, interval temporal logic are used to describe patterns with quite a lot of expressive power. The most representative interval logic if Helpern and Shoham Modal Logic of Allen's Relations, aka HS; recently, a series of results in the fields of neuroesthetics, medicine, physiology, neurophysiology, and mechanical engineering have been obtained based on this idea.

Summary

Symbolic modern artificial intelligence is based on a set of algorithms and methods that are designed to extract knowledge from data in the form of logical formulas. In the case of temporal data, in particular, interval temporal logic are used to describe patterns with quite a lot of expressive power. The most representative interval logic if Helpern and Shoham Modal Logic of Allen's Relations, aka HS; recently, a series of results in the fields of neuroesthetics, medicine, physiology, neurophysiology, and mechanical engineering have been obtained based on this idea. With the aim of improving the expressive power of the crisp version of such language(s), we introduced a fuzzy version of HS (FHS).

Summary

Symbolic modern artificial intelligence is based on a set of algorithms and methods that are designed to extract knowledge from data in the form of logical formulas. In the case of temporal data, in particular, interval temporal logic are used to describe patterns with quite a lot of expressive power. The most representative interval logic if Helpern and Shoham Modal Logic of Allen's Relations, aka HS; recently, a series of results in the fields of neuroesthetics, medicine, physiology, neurophysiology, and mechanical engineering have been obtained based on this idea. With the aim of improving the expressive power of the crisp version of such language(s), we introduced a fuzzy version of HS (FHS). Because FHS is a very natural and new logic, it makes sense to study classic problems for FHS, such as automatic reasoning with tableaux.

Fuzzy Modal Logic (1)

What is fuzzy (or many-valued) propositional logic?

Fuzzy Modal Logic (1)

What is fuzzy (or many-valued) propositional logic?

$\downarrow$

It is the generalization of propositional logic in which the Boolean 2-valued algebra is replaced by a richer truth value algebra. Typical choices include Heyting algebras and Łukasiewicz algebras

Fuzzy Modal Logic (1)

What is fuzzy (or many-valued) propositional logic?

↓

It is the generalization of propositional logic in which the Boolean 2-valued algebra is replaced by a richer truth value algebra. Typical choices include Heyting algebras and Łukasiewicz algebras

What is fuzzy (or many-valued) modal logic?

Fuzzy Modal Logic (1)

What is fuzzy (or many-valued) propositional logic?

↓

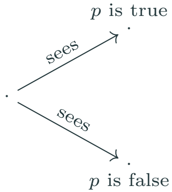
It is the generalization of propositional logic in which the
Boolean 2-valued algebra is replaced by a richer truth value algebra.
Typical choices include Heyting algebras and Łukasiewicz algebras

What is fuzzy (or many-valued) modal logic?

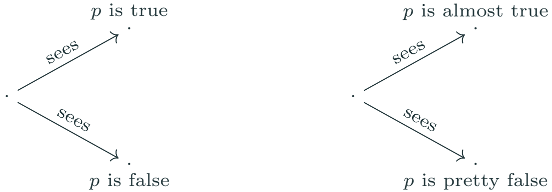
↓

It is modal logic in which the truth of
either accessibility relations, propositional letters on worlds, or both
is generalized as in the propositional case.
Many-valued modal logics have been introduced by Fitting, in 1991.

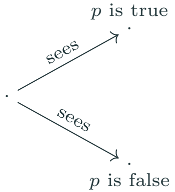
Fuzzy Modal Logic (2)

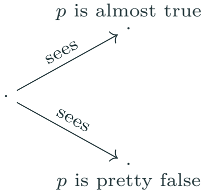


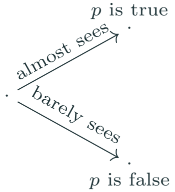
Fuzzy Modal Logic (2)



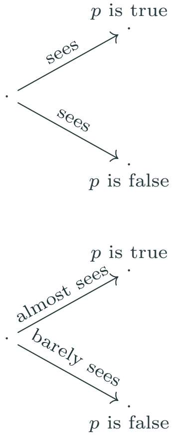
Fuzzy Modal Logic (2)



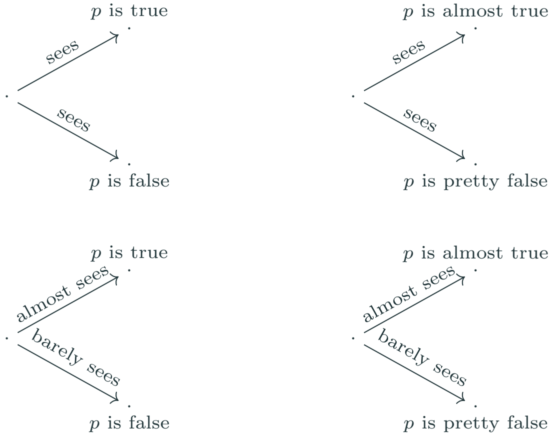




Fuzzy Modal Logic (2)

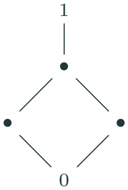


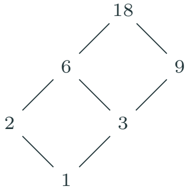
Fuzzy Modal Logic (2)

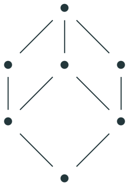


Fitting suggests using Heyting algebras towards a truly 'many-valued' semantics.

Heyting Algebras





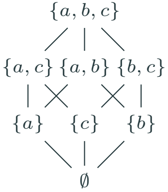


$\mathbb{G}_{\aleph_1}$

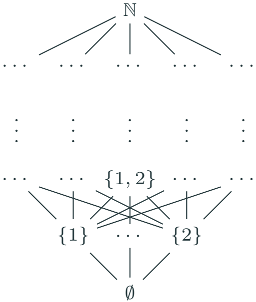
$\mathbb{G}_{\aleph_0}$

$\mathbb{G}_4$

$\mathbb{B}^1 = \mathbb{G}_2$

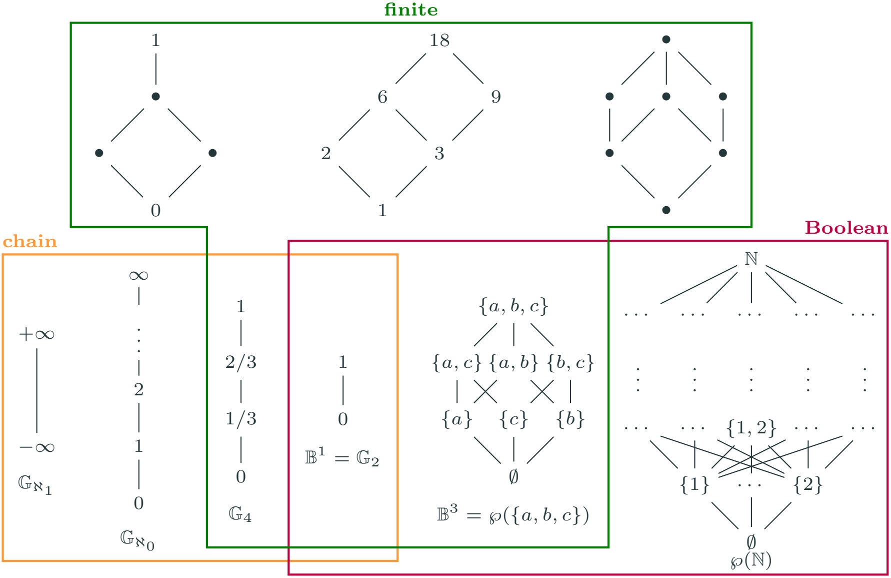


$\mathbb{B}^3 = \wp(\{a, b, c\})$

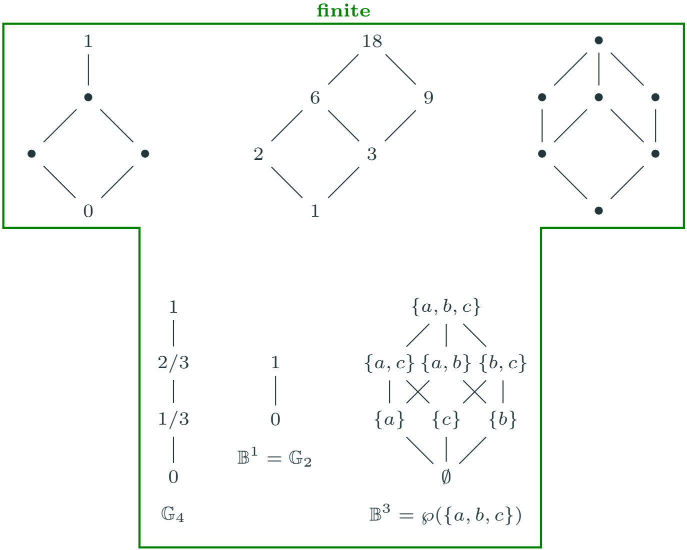


$\wp(\mathbb{N})$

Heyting Algebras



Heyting Algebras



Fuzzy HS in a Nutshell

Interval temporal logic comes in many forms. We consider here a specific variant called Halpern and Shoham's Modal Logic of Allen's Relations (HS).

Fuzzy HS in a Nutshell

Interval temporal logic comes in many forms. We consider here a specific variant called Halpern and Shoham's Modal Logic of Allen's Relations (HS). It is a modal logic based on a linearly ordered domain of which we consider every possible interval. Intervals are worlds. Such worlds are related by several differen accessibility relations, one for each Allen's relation (later, after, during, overlaps, begins, ends). A valuation function assigns the truth of every propositional letter to every interval.

Fuzzy HS in a Nutshell

Interval temporal logic comes in many forms. We consider here a specific variant called Halpern and Shoham's Modal Logic of Allen's Relations (HS). It is a modal logic based on a linearly ordered domain of which we consider every possible interval. Intervals are worlds. Such worlds are related by several differen accessibility relations, one for each Allen's relation (later, after, during, overlaps, begins, ends). A valuation function assigns the truth of every propositional letter to every interval. In its fuzzy version, FHS generalizes HS by using a finite Heyting algebra to soften both the degree of the relation between the current interval and the accessed one(s) and the truth value of propositional letters.

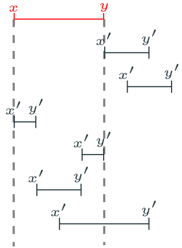
Fuzzy HS in a Nutshell

Interval temporal logic comes in many forms. We consider here a specific variant called Halpern and Shoham's Modal Logic of Allen's Relations (HS). It is a modal logic based on a linearly ordered domain of which we consider every possible interval. Intervals are worlds. Such worlds are related by several differen accessibility relations, one for each Allen's relation (later, after, during, overlaps, begins, ends). A valuation function assigns the truth of every propositional letter to every interval. In its fuzzy version, FHS generalizes HS by using a finite Heyting algebra to soften both the degree of the relation between the current interval and the accessed one(s) and the truth value of propositional letters. Satisfiability of HS formulas is undecidable over essentially every class of linearly ordered sets, and so is FHS.

Crisp HS (1)

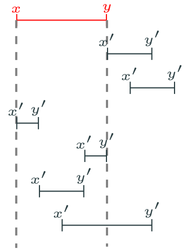
HS	Allen's relation
$\langle A \rangle$	$[x,y]R_A[x',y'] \Leftrightarrow y = x'$
$\langle L \rangle$	$[x,y]R_L[x',y'] \Leftrightarrow y < x'$
$\langle B \rangle$	$[x,y]R_B[x',y'] \Leftrightarrow x = x', y' < y$
$\langle E \rangle$	$[x,y]R_E[x',y'] \Leftrightarrow y = y', x < x'$
$\langle D \rangle$	$[x,y]R_D[x',y'] \Leftrightarrow x < x', y' < y$
$\langle O \rangle$	$[x,y]R_O[x',y'] \Leftrightarrow x < x' < y < y'$

Graphical repr.



Crisp HS (1)

HS	Allen's relation	Graphical repr.
$\langle A \rangle$	$[x,y]R_A[x',y'] \Leftrightarrow y = x'$	
$\langle L \rangle$	$[x,y]R_L[x',y'] \Leftrightarrow y < x'$	
$\langle B \rangle$	$[x,y]R_B[x',y'] \Leftrightarrow x = x', y' < y$	
$\langle E \rangle$	$[x,y]R_E[x',y'] \Leftrightarrow y = y', x < x'$	
$\langle D \rangle$	$[x,y]R_D[x',y'] \Leftrightarrow x < x', y' < y$	
$\langle O \rangle$	$[x,y]R_O[x',y'] \Leftrightarrow x < x' < y < y'$	



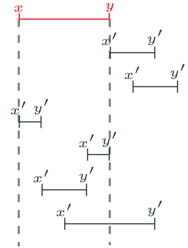
$$\varphi = p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle X \rangle \varphi$$
$$X \in \{A, L, B, E, D, O\}$$

$$M = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), V \rangle$$

Crisp HS (1)

HS	Allen's relation
$\langle A \rangle$	$[x,y]R_A[x',y'] \Leftrightarrow y = x'$
$\langle L \rangle$	$[x,y]R_L[x',y'] \Leftrightarrow y < x'$
$\langle B \rangle$	$[x,y]R_B[x',y'] \Leftrightarrow x = x', y' < y$
$\langle E \rangle$	$[x,y]R_E[x',y'] \Leftrightarrow y = y', x < x'$
$\langle D \rangle$	$[x,y]R_D[x',y'] \Leftrightarrow x < x', y' < y$
$\langle O \rangle$	$[x,y]R_O[x',y'] \Leftrightarrow x < x' < y < y'$

Graphical repr.



$$\varphi = p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle X \rangle \varphi$$
$$X \in \{A, L, B, E, D, O\}$$

$$M = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), V \rangle$$

$M, [x,y] \Vdash p$	iff	$p \in V([x,y])$
$M, [x,y] \Vdash \neg\varphi$	iff	$M, [x,y] \not\Vdash \varphi$
$M, [x,y] \Vdash \varphi \wedge \psi$	iff	$M, [x,y] \Vdash \varphi$ and $M, [x,y] \Vdash \psi$
$M, [x,y] \Vdash \langle X \rangle \varphi$	iff	for some$[z,t]$ s.t. $[x,y]R_X[z,t]$ $M, [z,t] \Vdash \varphi$

Crisp HS (2)

In crisp HS we may express sentences such as:

the fever spiked to over 39.5 during an episode of headache of level greater than 3

Crisp HS (2)

In crisp HS we may express sentences such as:
the fever spiked to over 39.5 during an episode of headache of level greater than 3

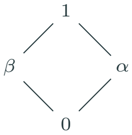
$$\downarrow$$

$$(head \geq 3) \wedge \langle D \rangle (max(fever) \geq 39.5)$$

Fuzzy HS (1)

To define FHS we fix an algebra
$\mathcal{H} = (H, \cap, \cup, \hookrightarrow, 0, 1)$,
and we proceed as follows:

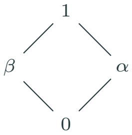
$\mathcal{H}$:



Fuzzy HS (1)

To define FHS we fix an algebra
$\mathcal{H} = (H, \cap, \cup, \hookrightarrow, 0, 1)$,
and we proceed as follows:

$\mathcal{H}$:

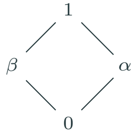


We consider a linear order $\mathbb{D}$ and we define *fuzzy* $\widetilde{<}, \widetilde{=}$ on top of $<$ with values in $\mathcal{H}$ - - - - - - - -> We axiomatize *adequate fuzzy linear orders*

Fuzzy HS (1)

To define FHS we fix an algebra
$\mathcal{H} = (H, \cap, \cup, \hookrightarrow, 0, 1)$,
and we proceed as follows:

$\mathcal{H}$:



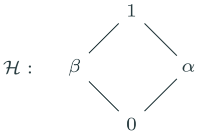
We consider a linear order $\mathbb{D}$ and we define *fuzzy* $\widetilde{<}, \widetilde{=}$ on top of $<$ with values in $\mathcal{H}$ - - - - - - - → We axiomatize *adequate fuzzy linear orders*

↓

We generalize the definition of all Allen's relations R_X - - - - - - - - - - - - - - → We get *fuzzy* relations $R_{\tilde{X}}$

Fuzzy HS (1)

To define FHS we fix an algebra
$\mathcal{H} = (H, \cap, \cup, \hookrightarrow, 0, 1)$,
and we proceed as follows:



We consider a linear order $\mathbb{D}$ and we define *fuzzy* $\widetilde{<}, \widetilde{=}$ on top of $<$ with values in $\mathcal{H}$ - - - - - - - → We axiomatize *adequate fuzzy linear orders*

↓

We generalize the definition of all Allen's relations R_X - - - - - - - - - - - - - - → We get *fuzzy* relations $R_{\tilde{X}}$

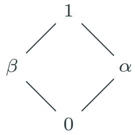
↓

We generalize the definition of interval $[x, y]$ using $\widetilde{<}$ - - - - - - - - - - - - - - - - - - → $[x, y]$ means $x \widetilde{<} y$

Fuzzy HS (1)

To define FHS we fix an algebra
$\mathcal{H} = (H, \cap, \cup, \hookrightarrow, 0, 1)$,
and we proceed as follows:

$\mathcal{H}$:



We consider a linear order $\mathbb{D}$ and we define *fuzzy* $\widetilde{<}, \widetilde{=}$ on top of $<$ with values in $\mathcal{H}$ ⇢ We axiomatize *adequate fuzzy linear orders*

↓

We generalize the definition of all Allen's relations R_X ⇢ We get *fuzzy* relations $R_{\tilde{X}}$

↓

We generalize the definition of interval $[x, y]$ using $\widetilde{<}$ ⇢ $[x, y]$ means $x \widetilde{<} y$

↓

We generalize the valuation function with values in $\mathcal{H}$ ⇢ $V(p) \in \mathcal{H}$

Fuzzy HS (2)

FHS	Allen's relation
$\langle A \rangle$	$[x, y]\widetilde{R}_A[x', y'] \Leftrightarrow y \widetilde{=} x'$
$\langle L \rangle$	$[x, y]\widetilde{R}_L[x', y'] \Leftrightarrow y \widetilde{<} x'$
$\langle B \rangle$	$[x, y]\widetilde{R}_B[x', y'] \Leftrightarrow x \widetilde{=} x', y' \widetilde{<} y$
$\langle E \rangle$	$[x, y]\widetilde{R}_E[x', y'] \Leftrightarrow y \widetilde{=} y', x \widetilde{<} x'$
$\langle D \rangle$	$[x, y]\widetilde{R}_D[x', y'] \Leftrightarrow x \widetilde{<} x', y' \widetilde{<} y$
$\langle O \rangle$	$[x, y]\widetilde{R}_O[x', y'] \Leftrightarrow x \widetilde{<} x' \widetilde{<} y \widetilde{<} y'$

Fuzzy HS (2)

FHS	Allen's relation
$\langle A\rangle$	$[x,y]\widetilde{R}_A[x',y'] \Leftrightarrow y \widetilde{=} x'$
$\langle L\rangle$	$[x,y]\widetilde{R}_L[x',y'] \Leftrightarrow y \widetilde{<} x'$
$\langle B\rangle$	$[x,y]\widetilde{R}_B[x',y'] \Leftrightarrow x \widetilde{=} x', y' \widetilde{<} y$
$\langle E\rangle$	$[x,y]\widetilde{R}_E[x',y'] \Leftrightarrow y \widetilde{=} y', x \widetilde{<} x'$
$\langle D\rangle$	$[x,y]\widetilde{R}_D[x',y'] \Leftrightarrow x \widetilde{<} x', y' \widetilde{<} y$
$\langle O\rangle$	$[x,y]\widetilde{R}_O[x',y'] \Leftrightarrow x \widetilde{<} x' \widetilde{<} y \widetilde{<} y'$

$$\varphi ::= \alpha \mid p \mid \varphi \vee \psi \mid \varphi \wedge \psi$$
$$\mid \varphi \rightarrow \psi \mid \langle X\rangle\varphi \mid [X]\varphi$$
$$X \in \{A, L, B, E, D, O\}$$

$$\widetilde{M} = \langle \mathbb{I}(\widetilde{\mathbb{D}}), \widetilde{V}\rangle$$
$$\widetilde{V} : \mathcal{P} \times \mathbb{I}(\widetilde{\mathbb{D}}) \mapsto H$$

Fuzzy HS (2)

FHS	Allen's relation
$\langle A\rangle$	$[x,y]\widetilde{R}_A[x',y'] \Leftrightarrow y\widetilde{=}x'$
$\langle L\rangle$	$[x,y]\widetilde{R}_L[x',y'] \Leftrightarrow y\widetilde{<}x'$
$\langle B\rangle$	$[x,y]\widetilde{R}_B[x',y'] \Leftrightarrow x\widetilde{=}x', y'\widetilde{<}y$
$\langle E\rangle$	$[x,y]\widetilde{R}_E[x',y'] \Leftrightarrow y\widetilde{=}y', x\widetilde{<}x'$
$\langle D\rangle$	$[x,y]\widetilde{R}_D[x',y'] \Leftrightarrow x\widetilde{<}x', y'\widetilde{<}y$
$\langle O\rangle$	$[x,y]\widetilde{R}_O[x',y'] \Leftrightarrow x\widetilde{<}x'\widetilde{<}y\widetilde{<}y'$

$$\varphi ::= \alpha \mid p \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \langle X\rangle\varphi \mid [X]\varphi$$
$$X \in \{A, L, B, E, D, O\}$$

$$\widetilde{M} = \langle \mathbb{I}(\widetilde{\mathbb{D}}), \widetilde{V}\rangle$$
$$\widetilde{V} : \mathcal{P} \times \mathbb{I}(\widetilde{\mathbb{D}}) \mapsto H$$

$$\begin{aligned}
\widetilde{V}(\alpha,[x,y]) &= \alpha,\\
\widetilde{V}(\varphi\wedge\psi,[x,y]) &= \widetilde{V}(\varphi,[x,y]) \cap \widetilde{V}(\psi,[x,y]),\\
\widetilde{V}(\varphi\vee\psi,[x,y]) &= \widetilde{V}(\varphi,[x,y]) \cup \widetilde{V}(\psi,[x,y]),\\
\widetilde{V}(\varphi\rightarrow\psi,[x,y]) &= \widetilde{V}(\varphi,[x,y]) \hookrightarrow \widetilde{V}(\psi,[x,y]),\\
\widetilde{V}(\langle X\rangle\varphi,[x,y]) &= \bigcup\{\widetilde{R}_X([x,y],[z,t]) \cap \widetilde{V}(\varphi,[z,t])\},\\
\widetilde{V}([X]\varphi,[x,y]) &= \bigcap\{\widetilde{R}_X([x,y],[z,t]) \hookrightarrow \widetilde{V}(\varphi,[z,t])\}.
\end{aligned}$$

Fuzzy HS (4)

In FHS we ask questions such as:
how true is that
the fever spiked to a pretty high value more or less during an episode of severe headache?

Fuzzy HS (4)

In FHS we ask questions such as:
how true is that
the fever spiked to a pretty high value more or less during an episode of severe headache?

$$\downarrow$$

$$(head \geq 3) \wedge \langle D \rangle (max(fever) \geq 39.5)$$

Fuzzy HS (4)

In FHS we ask questions such as:
how true is that
the fever spiked to a pretty high value more or less during an episode of severe headache?

$$\downarrow$$

$$(head \geq 3) \wedge \langle D \rangle (max(fever) \geq 39.5)$$

The satisfiability problem for FHS is stated as follows:
given a formula φ, is there a model $\widetilde{M}$ and an interval $[x, y]$ such that $\widetilde{V}(\varphi, [x, y]) \succ 0$?

A Tableau System for FHS: Overview

We designed a tableau system for FHS to answer the more general question: given φ and α, is there a model $\widetilde{M}$ such that $\widetilde{V}(\varphi, [x, y]) \succeq \alpha$ for some interval $[x, y]$?

A Tableau System for FHS: Overview

We designed a tableau system for FHS to answer the more general question: given φ and α, is there a model $\widetilde{M}$ such that $\widetilde{V}(\varphi, [x, y]) \succeq \alpha$ for some interval $[x, y]$? We focus on finite Heyting algebras. As a classic tableau system, in this case too we explore all consequences of asserting something in a generic linear model and over a generic interval.

A Tableau System for FHS: Overview

We designed a tableau system for FHS to answer the more general question: given φ and α, is there a model $\widetilde{M}$ such that $\widetilde{V}(\varphi, [x, y]) \succeq \alpha$ for some interval $[x, y]$? We focus on finite Heyting algebras. As a classic tableau system, in this case too we explore all consequences of asserting something in a generic linear model and over a generic interval. Unlike classic tableau system, however, such consequences have to take into account all possible combination of values of the algebra, and all possible placing for points on the (tentative) linear model.

A Tableau System for FHS: Overview

We designed a tableau system for FHS to answer the more general question: given φ and α, is there a model $\widetilde{M}$ such that $\widetilde{V}(\varphi, [x, y]) \succeq \alpha$ for some interval $[x, y]$? We focus on finite Heyting algebras. As a classic tableau system, in this case too we explore all consequences of asserting something in a generic linear model and over a generic interval. Unlike classic tableau system, however, such consequences have to take into account all possible combination of values of the algebra, and all possible placing for points on the (tentative) linear model. Alongside the logical part of the tableau, we have to maintain a algebraic one, in particular for exploring how points can be placed with respect to one another.

A Tableau System for FHS: Rules (1)

A rule can be of one of four types:

A Tableau System for FHS: Rules (1)

A rule can be of one of four types:

- $T(\alpha \to \varphi, [x, y])$ (it is true that in the tentative model φ is evaluated more than or equal to α on the interval $[x, y]$),
- $F(\alpha \to \varphi, [x, y])$ (it is false that in the tentative model φ is evaluated more than or equal to α on the interval $[x, y]$),
- $T(\varphi \to \alpha, [x, y])$ (it is true that in the tentative model φ is evaluated less than or equal to α on the interval $[x, y]$), and
- $F(\varphi \to \alpha, [x, y])$ (it is false that in the tentative model φ is evaluated less than or equal to α on the interval $[x, y]$).

A Tableau System for FHS: Rules (1)

A rule can be of one of four types:

- $T(\alpha \to \varphi, [x, y])$ (it is true that in the tentative model φ is evaluated more than or equal to α on the interval $[x, y]$),
- $F(\alpha \to \varphi, [x, y])$ (it is false that in the tentative model φ is evaluated more than or equal to α on the interval $[x, y]$),
- $T(\varphi \to \alpha, [x, y])$ (it is true that in the tentative model φ is evaluated less than or equal to α on the interval $[x, y]$), and
- $F(\varphi \to \alpha, [x, y])$ (it is false that in the tentative model φ is evaluated less than or equal to α on the interval $[x, y]$).

The judgments T and F are not symmetric, because a finite algebra may not be linearly ordered.

A Tableau System for FHS: Rules (1)

A rule can be of one of four types:

- $T(\alpha \to \varphi, [x, y])$ (it is true that in the tentative model φ is evaluated more than or equal to α on the interval $[x, y]$),
- $F(\alpha \to \varphi, [x, y])$ (it is false that in the tentative model φ is evaluated more than or equal to α on the interval $[x, y]$),
- $T(\varphi \to \alpha, [x, y])$ (it is true that in the tentative model φ is evaluated less than or equal to α on the interval $[x, y]$), and
- $F(\varphi \to \alpha, [x, y])$ (it is false that in the tentative model φ is evaluated less than or equal to α on the interval $[x, y]$).

The judgments T and F are not symmetric, because a finite algebra may not be linearly ordered. Applying a rule entails taking into account the fuzzy linear order on which the tentative model is based. At the beginning, we have only two points that form only one interval: $x \widetilde{<} y$. As the tableau grows, every branch is associated to a partially specified fuzzy linear order.

A Tableau System for FHS: Rules (2)

$$(T \succeq)\ \frac{T(\alpha \rightarrow \psi, [x,y], C)}{F(\psi \rightarrow \gamma, [x,y], c(B))}$$

where $\alpha \neq 0$ and γ is any maximal element not above α, i.e., $\gamma \not\succeq \alpha$

$$(F \succeq)\ \frac{F(\alpha \rightarrow \psi, [x,y], C)}{T(\psi \rightarrow \beta_i, [x,y], c(B)) \mid \ldots \mid T(\psi \rightarrow \beta_n, [x,y], c(B))}$$

where $\alpha \neq 0$ and $\beta_1, \ldots, \beta_n$ are all maximal elements not above α, i.e., $\beta_1, \ldots, \beta_n \not\succeq \alpha$

(a) Reverse rules (examples).

A Tableau System for FHS: Rules (2)

$$(T\succeq)\ \frac{T(\alpha \to \psi, [x,y], C)}{F(\psi \to \gamma, [x,y], c(B))}$$

where $\alpha \neq 0$ and γ is any maximal element not above α, i.e., $\gamma \not\succeq \alpha$

$$(F\succeq)\ \frac{F(\alpha \to \psi, [x,y], C)}{T(\psi \to \beta_i, [x,y], c(B)) \mid \ldots \mid T(\psi \to \beta_n, [x,y], c(B))}$$

where $\alpha \neq 0$ and $\beta_1, \ldots, \beta_n$ are all maximal elements not above α, i.e., $\beta_1, \ldots, \beta_n \not\succeq \alpha$

(a) Reverse rules (examples).

$$(T\wedge)\ \frac{T(\alpha \to (\psi \wedge \xi), [x,y], C)}{\begin{array}{c}T(\alpha \to \psi, [x,y], c(B))\\ T(\alpha \to \xi, [x,y], c(B))\end{array}}$$

where $\alpha \neq 0$

$$(F\wedge)\ \frac{F(\alpha \to (\psi \wedge \xi), [x,y], C)}{F(\alpha \to \psi, [x,y], c(B)) \mid F(\alpha \to \xi, [x,y], c(B))}$$

where $\alpha \neq 0$

(b) Propositional rules (examples).

A Tableau System for FHS: Rules (2)

$$(T\succeq)\ \frac{T(\alpha \to \psi, [x,y], C)}{F(\psi \to \gamma, [x,y], c(B))}$$

where $\alpha \neq 0$ and γ is any maximal element not above α, i.e., $\gamma \not\succeq \alpha$

$$(F\succeq)\ \frac{F(\alpha \to \psi, [x,y], C)}{T(\psi \to \beta_i, [x,y], c(B)) \mid \ldots \mid T(\psi \to \beta_n, [x,y], c(B))}$$

where $\alpha \neq 0$ and $\beta_1, \ldots, \beta_n$ are all maximal elements not above α, i.e., $\beta_1, \ldots, \beta_n \not\succeq \alpha$

(a) Reverse rules (examples).

$$(T\wedge)\ \frac{T(\alpha \to (\psi \wedge \xi), [x,y], C)}{\begin{array}{c}T(\alpha \to \psi, [x,y], c(B))\\ T(\alpha \to \xi, [x,y], c(B))\end{array}}$$

where $\alpha \neq 0$

$$(F\wedge)\ \frac{F(\alpha \to (\psi \wedge \xi), [x,y], C)}{F(\alpha \to \psi, [x,y], c(B)) \mid F(\alpha \to \xi, [x,y], c(B))}$$

where $\alpha \neq 0$

(b) Propositional rules (examples).

$$(T\Box)\ \frac{T(\alpha \to [X]\psi, [x,y], C)}{\begin{array}{c}T((\alpha \cap \beta_1) \to \psi, [z_1, t_1], c(B))\\ \ldots\\ T((\alpha \cap \beta_n) \to \psi, [z_n, t_n], c(B))\\ T(\alpha \to [X]\psi, [x,y], c(B))\end{array}}$$

where $\beta_i = R_X([x,y],[z_i,t_i])$, $[z_i,t_i] \in o(c(B))$, $\beta_i \succ 0$, and $\alpha \cap \beta_i \neq 0$

$$(T\Diamond)\ \frac{T(\langle X\rangle\psi \to \alpha, [x,y], C)}{\begin{array}{c}T((\psi \to (\beta_1 \hookrightarrow \alpha), [z_1, t_1], c(B))\\ \ldots\\ T(\psi \to (\beta_n \hookrightarrow \alpha), [z_n, t_n], c(B))\\ T(\langle X\rangle\psi \to \alpha, [x,y], c(B))\end{array}}$$

where $\beta_i = R_X([x,y],[z_i,t_i])$, $[z_i,t_i] \in o(c(B))$, $\beta_i \succ 0$, and $\beta_i \hookrightarrow \alpha \neq 1$

(c) Temporal rules (examples).

A Tableau System for FHS: Tableau (1)

Given an FHS formula φ and a finite Heyting algebra $\mathcal{H}$, the tableau τ for φ and $\alpha \in \mathcal{H}$ is an object of the type

$$\tau = (\mathcal{V}, \mathcal{E}, d, f, c),$$

where $(\mathcal{V}, \mathcal{E})$ is a tree with vertices (or nodes) in $\mathcal{V}$ and edges in $\mathcal{E}$.

A Tableau System for FHS: Tableau (1)

Given an FHS formula φ and a finite Heyting algebra $\mathcal{H}$, the tableau τ for φ and $\alpha \in \mathcal{H}$ is an object of the type

$$\tau = (\mathcal{V}, \mathcal{E}, d, f, c),$$

where $(\mathcal{V}, \mathcal{E})$ is a tree with vertices (or nodes) in $\mathcal{V}$ and edges in $\mathcal{E}$. The nodes in τ are partially ordered by the relation $\lhd$ (induced by the edges) and whose set of branches is denoted by $\mathcal{B}$,

$$d : \mathcal{V} \rightarrow \mathcal{D},$$

is a node labeling function, which associates a decoration $Q(\psi \rightarrow \alpha, [x, y], C)$ or $Q(\alpha \rightarrow \psi, [x, y], C)$ to any node ν, where $\psi \in sub(\varphi)$ and $x, y \in C$,

A Tableau System for FHS: Tableau (1)

Given an FHS formula φ and a finite Heyting algebra $\mathcal{H}$, the tableau τ for φ and $\alpha \in \mathcal{H}$ is an object of the type

$$\tau = (\mathcal{V}, \mathcal{E}, d, f, c),$$

where $(\mathcal{V}, \mathcal{E})$ is a tree with vertices (or nodes) in $\mathcal{V}$ and edges in $\mathcal{E}$. The nodes in τ are partially ordered by the relation $\lhd$ (induced by the edges) and whose set of branches is denoted by $\mathcal{B}$,

$$d : \mathcal{V} \to \mathcal{D},$$

is a node labeling function, which associates a decoration $Q(\psi \to \alpha, [x, y], C)$ or $Q(\alpha \to \psi, [x, y], C)$ to any node ν, where $\psi \in sub(\varphi)$ and $x, y \in C$, and

$$f : \mathcal{V} \to \{0, 1\}$$

is a node flag function, which determines which nodes have been already expanded,

A Tableau System for FHS: Tableau (1)

Given an FHS formula φ and a finite Heyting algebra $\mathcal{H}$, the tableau τ for φ and $\alpha \in \mathcal{H}$ is an object of the type

$$\tau = (\mathcal{V}, \mathcal{E}, d, f, c),$$

where $(\mathcal{V}, \mathcal{E})$ is a tree with vertices (or nodes) in $\mathcal{V}$ and edges in $\mathcal{E}$. The nodes in τ are partially ordered by the relation $\lhd$ (induced by the edges) and whose set of branches is denoted by $\mathcal{B}$,

$$d : \mathcal{V} \to \mathcal{D},$$

is a node labeling function, which associates a decoration $Q(\psi \to \alpha, [x, y], C)$ or $Q(\alpha \to \psi, [x, y], C)$ to any node ν, where $\psi \in sub(\varphi)$ and $x, y \in C$, and

$$f : \mathcal{V} \to \{0, 1\}$$

is a node flag function, which determines which nodes have been already expanded,

$$c : \mathcal{B} \to \mathcal{C}$$

is a branch labeling function, which associates every branch to the constraint system in the decoration of its leaf,

A Tableau System for FHS: Tableau (1)

Given an FHS formula φ and a finite Heyting algebra $\mathcal{H}$, the tableau τ for φ and $\alpha \in \mathcal{H}$ is an object of the type

$$\tau = (\mathcal{V}, \mathcal{E}, d, f, c),$$

where $(\mathcal{V}, \mathcal{E})$ is a tree with vertices (or nodes) in $\mathcal{V}$ and edges in $\mathcal{E}$. The nodes in τ are partially ordered by the relation $\lhd$ (induced by the edges) and whose set of branches is denoted by $\mathcal{B}$,

$$d : \mathcal{V} \to \mathcal{D},$$

is a node labeling function, which associates a decoration $Q(\psi \to \alpha, [x, y], C)$ or $Q(\alpha \to \psi, [x, y], C)$ to any node ν, where $\psi \in sub(\varphi)$ and $x, y \in C$, and

$$f : \mathcal{V} \to \{0, 1\}$$

is a node flag function, which determines which nodes have been already expanded,

$$c : \mathcal{B} \to \mathcal{C}$$

is a branch labeling function, which associates every branch to the constraint system in the decoration of its leaf, and it has been obtained starting from the initial tableau τ_0

$$(\{\nu_0\}, \emptyset, \{(\nu_0, T(\alpha \to \varphi, [x, y], \{x, y, \widetilde{<}(x, y) \succ 0\}))\}, \{(\nu_0, 0)\}, \{(\nu_0, \{x, y, \widetilde{<}(x, y) \succ 0\})\})$$

by iteratively applying the branch expansion rule to the closest-to-the-root node ν such that $f(\nu) = 0$ and every leaf ν' such that $\nu \lhd \nu'$, until no further application is possible or all branches have been closed. The tableau is closed (resp., open) if all its branches (resp., at least one of its branches) are (resp., is) closed ✗ by some condition (resp., open ✓).

Lemma 1 (soundness).

Let φ be an FHS formula and $\alpha \in \mathcal{H}$ a constant of a finite Heyting algebra. Then, if φ is α-satisfiable, then the tableau τ for φ and α is open.

A Tableau System for FHS: Tableau (2)

Lemma 1 (soundness).

Let φ be an FHS formula and $\alpha \in \mathcal{H}$ a constant of a finite Heyting algebra. Then, if φ is α-satisfiable, then the tableau τ for φ and α is open.

Lemma 2 (completeness).

Let φ be an FHS formula and $\alpha \in \mathcal{H}$ a constant of a finite Heyting algebra. If τ is an open tableau for φ and α, then φ is α-satisfiable.

A Tableau System for FHS: Tableau (2)

Lemma 1 (soundness).

Let φ be an FHS formula and $\alpha \in \mathcal{H}$ a constant of a finite Heyting algebra. Then, if φ is α-satisfiable, then the tableau τ for φ and α is open.

Lemma 2 (completeness).

Let φ be an FHS formula and $\alpha \in \mathcal{H}$ a constant of a finite Heyting algebra. If τ is an open tableau for φ and α, then φ is α-satisfiable.

Theorem 3 (semi-decision procedure).

The tableau system for FHS is sound and complete. Moreover, it is also a semi-decision procedure in the case of finite domains.

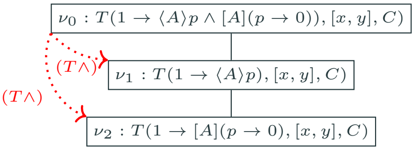
A Tableau System for FHS: Example

$\nu_0 : T(1 \to \langle A \rangle p \wedge [A](p \to 0)), [x, y], C)$

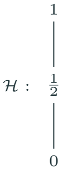
$C = \{\widetilde{<}(x, y) \succ 0\}$

$\mathcal{H}$: 1 — $\frac{1}{2}$ — 0

A Tableau System for FHS: Example



$C = \{\widetilde{<}(x, y) \succ 0\}$



A Tableau System for FHS: Example

$\nu_0 : T(1 \to \langle A \rangle p \wedge [A](p \to 0)), [x, y], C)$

$(T\wedge)$ $\nu_1 : T(1 \to \langle A \rangle p), [x, y], C)$

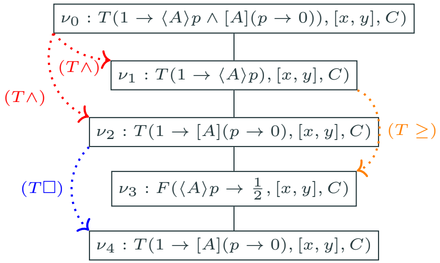
$(T\wedge)$ $\nu_2 : T(1 \to [A](p \to 0), [x, y], C)$ $(T \geq)$

$\nu_3 : F(\langle A \rangle p \to \frac{1}{2}, [x, y], C)$

$C = \{\widetilde{<}(x, y) \succ 0\}$

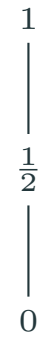
$\mathcal{H} :$ 1 — $\frac{1}{2}$ — 0

A Tableau System for FHS: Example

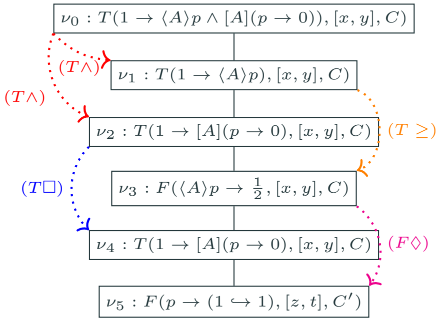


$C = \{\widetilde{<}(x,y) \succ 0\}$

$\mathcal{H}:$



A Tableau System for FHS: Example

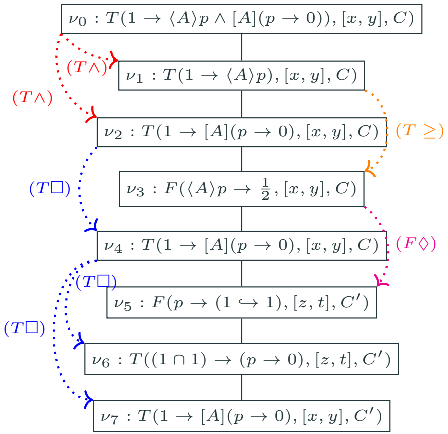


$C = \{\widetilde{<}(x, y) \succ 0\}$

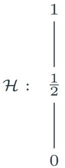
$\mathcal{H}$: 1 — $\frac{1}{2}$ — 0

$C' = C \cup \{\widetilde{\cong}(x, y) = 0, \widetilde{<}(z, t) \succ 0, \widetilde{\cong}(y, z) = 1\}$

A Tableau System for FHS: Example

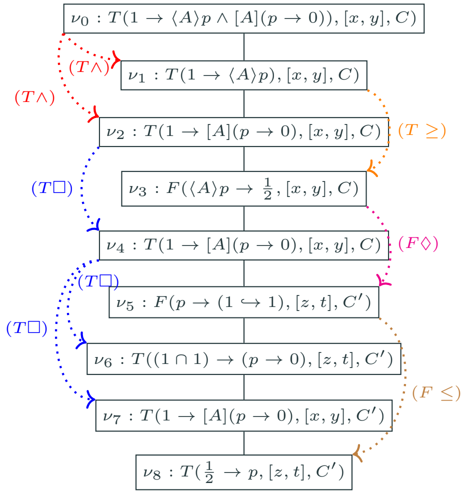


$$C = \{\widetilde{<}(x, y) \succ 0\}$$



$$C' = C \cup \{\widetilde{\cong}(x, y) = 0, \widetilde{<}(z, t) \succ 0, \widetilde{\cong}(y, z) = 1\}$$

A Tableau System for FHS: Example

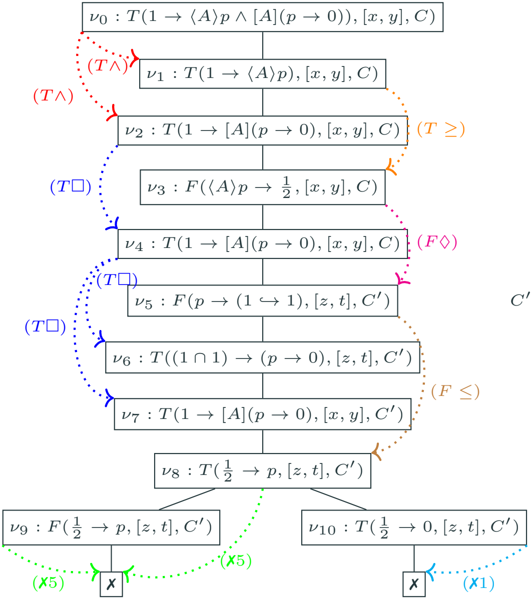


$C = \{\widetilde{<}(x,y) \succ 0\}$

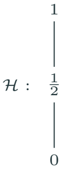
$\mathcal{H}$: 1 — $\frac{1}{2}$ — 0

$C' = C \cup \{\widetilde{\equiv}(x,y) = 0, \widetilde{<}(z,t) \succ 0, \widetilde{\equiv}(y,z) = 1\}$

A Tableau System for FHS: Example



$C = \{\widetilde{<}(x,y) \succ 0\}$



$C' = C \cup \{\widetilde{\equiv}(x,y) = 0, \widetilde{<}(z,t) \succ 0, \widetilde{\equiv}(y,z) = 1\}$

Conclusions

Tableau systems for many valued modal logics were introduced by Fitting, and we adapted the original proposal to the case of FHS.

.

Conclusions

Tableau systems for many valued modal logics were introduced by Fitting, and we adapted the original proposal to the case of FHS. Even in the crisp case, implementations of tableau systems for interval temporal logics, especially HS-like, are virtually non-existing. Experiments have shown that most clever solutions that have been applied for modal logics and even for point-based temporal logic may not be effective in the interval case .

Conclusions

Tableau systems for many valued modal logics were introduced by Fitting, and we adapted the original proposal to the case of FHS. Even in the crisp case, implementations of tableau systems for interval temporal logics, especially HS-like, are virtually non-existing. Experiments have shown that most clever solutions that have been applied for modal logics and even for point-based temporal logic may not be effective in the interval case . As part of a larger project focused on modal symbolic learning, especially with (variants of) HS from real-world data, our exploration of a fuzzy counterpart of our solutions led us to introduce FHS and, now, study deduction systems such as tableaux.

Conclusions

Tableau systems for many valued modal logics were introduced by Fitting, and we adapted the original proposal to the case of FHS. Even in the crisp case, implementations of tableau systems for interval temporal logics, especially HS-like, are virtually non-existing. Experiments have shown that most clever solutions that have been applied for modal logics and even for point-based temporal logic may not be effective in the interval case . As part of a larger project focused on modal symbolic learning, especially with (variants of) HS from real-world data, our exploration of a fuzzy counterpart of our solutions led us to introduce FHS and, now, study deduction systems such as tableaux. In the future, we plan to incorporate FHS in our general purpose learning systems for non-tabular data, and for temporal data in particular, and this will include a realization of an efficient implementation of the tableau system.